Groupschemes

Christian Dahlhausen

Notes typed by
Christian Merten

WiSe 2023

Contents

Chapter 1

Introduction

Literature: Görtz-Wedhorn: Algebraic Geometry I and II

The goal of this lecture is a brief introduction to the theory of group schemes.

Definition 1.1 (Group object)**.** Let $\mathcal{C}$ be a category with finite products. A *group object in* $\mathcal{C}$ is the data (G, m, e, i) where

- G is an object of $\mathcal{C}$
- $m\colon G \times G \to G$ is the multiplication map
- $e\colon 1 \to G$ is the unit
- $i\colon G \to G$ is the inversion map

such that the following diagrams commute

$$
\begin{tikzcd}
G \times G \times G \arrow[r, "m\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times m"] & G \times G \arrow[d, "m"] \\
G \times G \arrow[r, "m"] & G
\end{tikzcd}
,\quad
\begin{tikzcd}
G \times G \arrow[r, "m"] & G \arrow[dl] \\
G \times 1 \arrow[u, "\mathrm{id}\times e"] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \arrow[r, "\mathrm{id}\times i"] \arrow[d] & G \times G \arrow[d, "m"] \\
1 \arrow[r, "e"] & G
\end{tikzcd}
.
$$

G is called *commutative*, if additionally the diagram

$$
\begin{tikzcd}
G \times G \arrow[r, "\mathrm{swap}"] \arrow[d, "m"] & G \times G \arrow[dl, "m"] \\
G &
\end{tikzcd}
$$

commutes.

A *morphism of group objects* $(G, m, e, i) \to (G', m', e', i')$ is a morphism $f\colon G \to G'$ in $\mathcal{C}$ such that the diagrams

$$
\begin{tikzcd}
G \times G \arrow[r, "f\times f"] \arrow[d, "m"] & G' \times G' \arrow[d, "m'"] \\
G \arrow[r, "f"] & G'
\end{tikzcd}
,\quad
\begin{tikzcd}
G \arrow[r, "f"] & G' \\
1 \arrow[u, "e"] \arrow[ur, "e'"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "i"] & G' \arrow[d, "i'"] \\
G \arrow[r, "f"] & G'
\end{tikzcd}
.
$$

This defines the category $\mathrm{Grp}(\mathcal{C})$ of group objects in $\mathcal{C}$.

Example 1.2.

- $\mathrm{Grp}(\mathrm{Set}) \simeq \mathrm{Grp}$
- $\mathrm{Grp}(\mathrm{Grp}) \simeq \mathrm{Ab}$
- $\mathrm{Grp}(\mathrm{Ab}) \simeq ?$

- $\mathrm{Grp}(Top) \simeq$ topological Groups
- $\mathrm{Grp}(Mfd_\infty) \simeq$ Lie Groups

Definition 1.3 (group scheme)**.** Let S be a scheme. An *S-group scheme* or an *S-group* is a group object in the category of schemes over S.

Remark 1.4. Let S be a scheme. The structure of a group scheme over S on a S-scheme G is equivalent to a factorisation of the functor of points

$$\begin{array}{ccc} \mathrm{Sch}_S & \longrightarrow & \mathrm{Set} \\ \downarrow & \nearrow & \\ \mathrm{Grp} & & \end{array}$$

via the forgetful functor from groups to sets.

Example 1.5. Let S be a scheme.

(i) Let Γ be a group. Then $G = \Gamma_S$ where $G(T) \coloneqq \{$ locally constant maps $T \to \Gamma$ $\}$

(ii) (additive group) $\mathbb{G}_{a,S}$ where $\mathbb{G}_{a,S}(T) = \mathcal{O}_T(T)$. We have $\mathbb{G}_{a,S} \simeq \mathbb{A}^1_S$.

(iii) (multiplicative group) $\mathbb{G}_{m,S}$ where $\mathbb{G}_{m,S}(T) \coloneqq \mathcal{O}_T(T)^\times$.

(iv) (roots of unity) $\mu_{n,S}$ ($n \geq 1$) where $\mu_{n,S}(T) = \{x \in \mathcal{O}_T(T)^\times \mid x^n = 1\}$.

(v) $S = \mathrm{Spec}(R)$. $\mathrm{GL}_{n,R} = \mathrm{Spec}(A)$ where $A = R[T_{ij} \mid 1 \leq i,j \leq n][\det^{-1}]$ where $\det = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) T_{1\sigma(1)} \cdots T_{n\sigma(n)}$. We obtain $\mathrm{GL}_{n,S}$ by base changing $\mathrm{GL}_{n,\mathbb{Z}}$.

Lemma 1.6. *Let G be a S-group. Then $G \to S$ is separated if and only if $S \xrightarrow{e} G$ is a closed immersion.*

Definition 1.7. Let R be a ring. A *(commutative) Hopf-Algebra* over R is a group object in $\mathrm{Alg}_R^{\mathrm{op}}$, where $\mathrm{Alg}_R = \mathrm{CRing}_R$.

Remark 1.8. For a R-Hopf-Algebra A, we denote the canonical maps by

- $\mu\colon A \to A \otimes_R A$ (Comultiplication)
- $\varepsilon\colon A \to R$ (Counit)
- $\iota\colon A \to A$ (Antipode)

A Hopf-Algebra is called *cocommutative*, if the associated group object in $\mathrm{Alg}_R^{\mathrm{op}}$ kommutativ ist.

Remark 1.9. For a ring R, by construction we have an equivalence of categories between the category of affine R-group schemes and the opposite category of R-Hopf-Algebras.

Example 1.10. The additive group $\mathbb{G}_{a,R} = \mathrm{Spec}(R[t])$ has

- comultiplication $\mu\colon R[t] \to R[t] \otimes_R R[t], t \mapsto t \otimes 1 - 1 \otimes t$.
- counit $\varepsilon\colon R[t] \to R, t \mapsto 0$
- antipode $\iota\colon R[t] \to R[t], t \mapsto -t$

Proof. For any R-algebra A we have $\mathbb{G}_{a,R}(A) = A$ and the diagram

$$\begin{array}{ccc} \mathbb{G}_{a,R}(A) \times \mathbb{G}_{a,R}(A) & \xrightarrow{m} & \mathbb{G}_{a,R}(A) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Hom}_R(R[s_1, s_2], A) & \xrightarrow{\mu^*} & \mathrm{Hom}_R(R[t], A) \end{array}.$$

□

Definition 1.11. Let G be a S-group. A *subgroupscheme* of G is a subscheme $H \subseteq G$ such that

1) for all $T \in \mathrm{Sch}_S$, we have $H(T) \subseteq G(T)$ a subgroup,
2) We have commutative diagrams

$$\begin{array}{ccccc} H \times_S H & \longrightarrow & G \times_S G & \xrightarrow{m} & G \\ \downarrow & & & \nearrow & \\ H & & & & \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \xrightarrow{e} & G \\ \downarrow & \nearrow & \\ H & & \end{array}$$

A subgroup scheme $H \subseteq G$ is *normal* if $H(T)$ is a normal subgroup of $G(T)$ for all $T \in \mathrm{Sch}_S$.

For a morphism $f\colon G \to G'$ of S-groups and a subgroup $H' \subseteq G'$, let $f^{-1}(H')$ be $G \times'_G H$. For $H' = 1 \xrightarrow{e} G'$, we obtain the *kernel of* f and the cartesian square

$$\begin{array}{ccc} \mathrm{Ker}(f) & \longrightarrow & G \\ \downarrow & & \downarrow f \\ S & \xrightarrow{e} & G' \end{array} .$$

Remark 1.12. The kernel of a homomorphism f of S-groups is for any S-scheme T given by

$$\mathrm{Ker}(f)(T) = \ker\left(f(T)\right).$$

In particular, the $\mathrm{Ker}(f)$ is normal.

Definition 1.13. Let G be a S-group, T a S-scheme and $g \in G(T) = \mathrm{Hom}_S(T, G)$. Define the *lefttranslation by* g as

$$\begin{array}{ccc} G_T & \xleftarrow{=} & T \times_T G_T \\ \downarrow t_g & & \downarrow g \times \mathrm{id} \\ G_T & \xleftarrow[m]{} & G_T \times_T G_T \end{array} .$$

Remark 1.14. In the situation of 1.13, for every $T' \xrightarrow{f} T$, the map

$$t_g(T')\colon G_T(T') = G(T') \longrightarrow G(T') = G_T(T')$$

is the lefttranslation by the element $f^*(g) \in G(T')$.

Remark 1.15. Consider

$$\begin{array}{ccc} G \times_S G & \xrightarrow{(g,h) \mapsto (gh,h)} & G \times_S G \\ \downarrow m & \swarrow \mathrm{pr}_1 & \\ G & & \end{array} .$$

Let $\mathcal{P}$ be a property of morphisms stable under base change and composition with isomorphisms. Then whenever $G \to S$ satisfies $\mathcal{P}$, then m satisfies $\mathcal{P}$.

1.1 Useful statements on schemes

Let k be a field.

Definition 1.16. Let $\mathcal{P}$ be a property of schemes over fields. For a k-scheme X we say X *is geometrically* $\mathcal{P}$ if for all field extensions K/k the base change $X_K \to \operatorname{Spec} K$ is $\mathcal{P}$.

Example 1.17. The $\mathbb{R}$-scheme $X = \operatorname{Spec}\left(\mathbb{R}[x]/(x^2+1)\right)$ is irreducible but not geometrically irreducible.

Proposition 1.18. *For a k-scheme X the following are equivalent:*

(i) X is geometrically reduced

(ii) for every reduced k-scheme Y, the fibre product $X \times_k Y$ is reduced.

(iii) X is reduced and for every generic point $\eta \in X$ of an irreducible component of X, the field extension $\kappa(\eta)/k$ is separable.

(iv) There exists a perfect field Ω and an extension Ω/k such that X_Ω is reduced.

(v) For all finite and purely inseparable field extensions K/k, the base change X_K is reduced.

Proof. Reducedness is a local property, so without loss of generality $X = \operatorname{Spec} A$. Moreover we may assume that X itself is reduced. Let $\{\eta_i\}_{i \in I}$ be the set of generic points of irreducible components of X. Then we obtain an inclusion

$$A \hookrightarrow \prod_{i \in I} \underbrace{\kappa(\eta_i)}_{=S_i^{-1}A} .$$

We claim that for any field extension L/k the ring $A \otimes_k L$ is reduced if and only if for all $i \in I$ the ring $\kappa(\eta_i) \otimes_k L$ is reduced.

proof of the claim. ($\Rightarrow$): follows since forming the nilradical commutes with localisations. ($\Leftarrow$): We have

$$A \otimes_k L \hookrightarrow \left(\prod_{i \in I} \kappa(\eta_i)\right) \otimes_k L \hookrightarrow \prod_{i \in I} \kappa(\eta_i) \otimes_k L.$$

□

The claim immediatly implies the equivalence of (iii), (iv), (v) and (1). Since (ii) trivially implies (i). It remains to show that (iii) implies (2). Without loss of generality we may take $Y = \operatorname{Spec} B$ and set $\{\lambda_j\}_{j \in J}$ to be the generic points of Y. Then we obtain

$$A \otimes_k B \hookrightarrow A \otimes_k \left(\prod_{j \in J} \kappa(\lambda_j)\right) \hookrightarrow \left(\prod_{i \in I} \kappa(\eta_i)\right) \otimes_k \left(\prod_{j \in J} \kappa(\lambda_j)\right) \hookrightarrow \prod_{i,j} \underbrace{\kappa(\eta_i) \otimes_k \kappa(\eta_j)}_{\text{reduced}} .$$

□

Corollary 1.19. *If k is perfect, then reduced and geometrically reduced are equivalent.*

Remark 1.20. The statements in 1.18 also hold when *reduced* is replaced by *irreducible* or *integral.*

Proposition 1.21. *Let $f\colon X \to Y$ be a morphism of schemes that is locally of finite presentation. Then f is open if and only if for every point $x \in X$ and every point $y' \in Y$ with $y = f(x) \in \overline{\{y'\}}$ there exists $x' \in X$ with $x \in \overline{\{x'\}}$ such that $f(x') = y'$.*

Proof. Assume $X = \text{Spec } B$ and $Y = \text{Spec } A$. ($\Rightarrow$): Then set

$$Z := \text{Spec } \mathcal{O}_{X,x} \cap \bigcap_{t \in B \setminus \mathfrak{p}_x} D(t).$$

Since f is open, $y' \in f(D(t))$ for all $t \in B \setminus \mathfrak{p}_x$. Set $f_t := f|_{D(t)}$. Then $f_t^{-1}(y') \neq \emptyset$. For sake of contradiction suppose that $y' \notin f(Z)$. Then set $g\colon \text{Spec } \mathcal{O}_{X,x} \to X \xrightarrow{f} Y$. Therefore

$$\emptyset = g^{-1}(y') = \text{Spec } \left(\mathcal{O}_{X,x} \otimes_A \kappa(y')\right).$$

Thus

$$0 = \mathcal{O}_{X,x} \otimes_A \kappa(y') = \text{colim}_{t \in B \setminus \mathfrak{p}_x} \underbrace{B_t \otimes_A \kappa(y')}_{\neq 0}$$

which is a contradiction.

($\Leftarrow$): Show $f(X) \subseteq Y$ is open. By Chevalley's theorem ([?], 10.70), the image $f(X)$ is constructible. In the noetherian case use that open is equivalent to constructible and stable under generalizations ([?], 10.17). In the general case write A as a colimit of noetherian rings and conclude by careful general nonsense. □

Lemma 1.22. *Let $f\colon X \to Y$ be flat, $x \in X$, $y = f(x)$, $y' \in Y$ a generalization of y. Then there exists a generalization x' of x such that $f(x') = y'$.*

Proof. Set $A = \mathcal{O}_{Y,y}$, $B = \mathcal{O}_{X,x}$ and $\varphi\colon A \to B$. Since $y \in \text{im}(f)$ we have $\mathfrak{m}_y B \neq B$ and B is faithfully flat A-module (since φ is local and flat). Thus

$$0 \neq B \otimes_A \kappa(y'),$$

i.e. $f^{-1}(y') \cap \text{Spec } B \neq \emptyset$. □

Corollary 1.23. *Let $f\colon X \to Y$ be flat and locally of finite presentation. Then f is universally open.*

Proof. From 1.21 and 1.22 follows that flat and locally of finite presentation implies open. Since the former two properties are stable under base change, the result follows. □

Corollary 1.24. *Let $f\colon X \to S$ be locally of finite presentation. If $|S|$ is discrete, then every morphism $X \to S$ is universally open.*

Definition 1.25. Let $f\colon X \to Y$. We say

(i) f is *flat in* $x \in X$ if $f_x^\#\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

(ii) f is *flat* if f is flat in every point.

Example 1.26. (1) $X \to \text{Spec } k$ is flat.

(2) $\mathbb{A}^n_Y \to Y$ and $\mathbb{P}^n_Y \to Y$ are flat.

(3) Let $f\colon Z \hookrightarrow Y$ be a closed immersion. Then f is flat and locally of finite presentation if and only if f is an open immersion.

Proposition 1.27. *The following holds*

(i) $\text{Spec } B \to \text{Spec } A$ *is flat if and only if* $A \to B$ *is flat.*

(ii) Flatness is stable under base change and composition.

(iii) Flatness is local on the source and the target.

(iv) *Open immersions are flat.*

(v) *A morphism* $f\colon X \to Y$ *is flat if and only if for every* $y \in Y$ *the canonical morphism*

$$X \times_Y \operatorname{Spec}(\mathcal{O}_{X,y}) \to \operatorname{Spec}(\mathcal{O}_{Y,y})$$

is flat.

Definition 1.28. A morphism $f\colon X \to Y$ is called *faithfully flat* if f is flat and surjective.

Example 1.29. $\operatorname{Spec} \overline{k} \to \operatorname{Spec} k$ is faithfully flat.

Lemma 1.30. *Let* $\mathcal{C}$ *be a category with equalizers,* $F\colon \mathcal{C} \to \mathcal{D}$ *a conservative (i.e. reflects isomorphisms) functor that commutes with equalizers. Then* F *is faithful.*

Proof. Left as an exercise to the reader. □

Proposition 1.31. *Is* $f\colon X \to Y$ *faithfully flat, then* $f^*\colon \operatorname{QCoh}(Y) \to \operatorname{QCoh}(X)$ *faithful.*

Proof. Can be deduced from 1.30. The details are left to the reader. □

Remark 1.32 (Faithfully flat descent)**.** The statement from 1.31 can be - from a carefully selected viewpoint - viewn as the statement that the functor $X \mapsto \operatorname{QCoh}(X)$ satisfies the sheaf condition for faithfully flat and quasicompact morphisms, i.e. that the diagram

$$\operatorname{QCoh}(Y) \xrightarrow{f^*} \operatorname{QCoh}(X) \overset{\mathrm{pr}_1^*}{\underset{\mathrm{pr}_2^*}{\rightrightarrows}} \operatorname{QCoh}(X \times_Y X) \Rrightarrow \underbrace{\operatorname{QCoh}(X \times_Y X \times_Y X)}_{\text{corresponds to the cocycle condition}}$$

is a limit diagram.

Proposition 1.33 ([?], 14.53)**.** *Let* $f\colon X \to Y$ *be a* S*-morphism and* $g\colon S' \to S$ *faithfully flat and quasicompact. Denote by* $f' = f \times_S S'$*. If* f' *is*

(i) *(locally) of finite type or (locally) of finite presentation,*

(ii) *isomorphism / monomorphism,*

(iii) *open / closed / quasicompact immersion,*

(iv) *proper / affine / finite,*

then f *has the same property.*

1.2 Regular Schemes over Fields

Remark 1.34. Coming from differential geometry, we have three notions of the tangent space of a manifold M at a point $x \in M$:

- $T_xM = \{\alpha : (-\varepsilon, \varepsilon) \to M \mid \varepsilon > 0,\ \alpha(0) = x\}/\text{change of charts}$
- $T_xM = \operatorname{Der}(\mathcal{O}_{M,x}, \mathbb{R})$
- $T_xM = \operatorname{Hom}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$

Remark 1.35. As a reminder: for a noetherian local ring $(A, \mathfrak{m})$ of dimension d, the following are equivalent:

- $\operatorname{gr}_{\mathfrak{m}}(A) \cong A/\mathfrak{m}[T_1, \ldots, T_d]$,
- $\dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) = d$,

- $\mathfrak{m}$ has a generator set of d elements.

In this case, A is called *regular*.

A regular ring will always be an integral domain.

Definition 1.36. A locally noetherian scheme X is called *regular in* $x \in X$ if $\mathcal{O}_{X,x}$ is a regular noetherian local ring. Write

$$X_{\text{reg}} := \{x \in X \mid X \text{ is regular in x}\} .$$

We call X *regular* if $X_{\text{reg}} = X$.

The *tangent space* of X in x is defined via

$$T_x M := \operatorname{Hom}_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2, \kappa(x)) .$$

Remark. If X is integral, then $\mathfrak{m}_\eta = 0$ and thus $T_\eta X = 0$.

Example 1.37. Let k be a field and $f_1, \dots, f_r \in k[T_1, \dots, T_n]$ polynomials. Set $X = V(f_1, \dots, f_r) \subseteq \mathbb{A}_k^n$. For $x \in \mathbb{A}_k^n(k)$ we have an isomorphism

$$k^n \to T_x \mathbb{A}_k^n, \quad (v_1, \dots, v_n) \mapsto (\overline{g} \mapsto \sum_i v_i \frac{\partial g}{\partial T_i}(x)) .$$

The map $k[S_1, \dots, S_r] \to k[T_1, \dots, T_n]$, $S_i \mapsto T_i$ induces morphisms $f : \mathbb{A}_k^n \to \mathbb{A}_k^r$ and $df_x : T_x \mathbb{A}_k^n \to T_{f(x)} \mathbb{A}_k^r$ which fits into the following diagram

$$\begin{array}{ccc} T_x \mathbb{A}_k^n & \xrightarrow{df_x} & T_{f(x)} \mathbb{A}_k^r \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ k^n & \xrightarrow{\cdot J(f)} & k^r. \end{array}$$

Here $J(f)$ denotes the Jacobian. Claim: $T_x X = \ker(df_x)$.

Definition 1.38. Set $k[\varepsilon] = k[X]/(X^2)$. For X/k and $x \in X(k)$ define $X(k[\varepsilon])_x$ as the pullback

$$\begin{array}{ccc} X(k[\varepsilon])_x & \hookrightarrow & X(k[\varepsilon]) \\ \downarrow & & \downarrow \\ \{x\} & \hookrightarrow & X(k). \end{array}$$

Proposition 1.39. *We have a bijection* $X(k[\varepsilon])_x \xrightarrow{\cong} T_x X$ *which is functorial in* (X, x).

Proof. Left as an exercise. □

Definition 1.40. Let $f : X \to Y$ be a morphism of schemes and $d \geq 0$. We call f *smooth of relative degree* d *in* $x \in X$ if there exist neighbourhoods $x \in U \subseteq X$ open, $f(x) \in \operatorname{Spec}(R) = V \subseteq Y$ open affine as well as an $n \geq 0$ and polynomials $f_1, \dots, f_{n-d} \in R[T_1, \dots, T_n]$ such that

$$\begin{array}{ccc} U & \xhookrightarrow{\text{open}} & \operatorname{Spec}(R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d})) \\ & {\scriptstyle f}\searrow & \downarrow \\ & & V \end{array}$$

commutes and $J_{f_1, \dots, f_{n-d}}(f) \in M_{n-d,n}(\kappa(x))$ is of full rank.

Call f *smooth of relative degree* d if this is the case everywhere.

Proposition 1.41 ([?],6.15)**.**

1. *If $f : X \to Y$ is smooth in $x \in X$, then f is smooth in an open neighbourhood of x.*
2. *Smoothness of relative dimension d is local on source and target. It is closed under base change and composition (where in the latter degree is additive).*
3. *Open immersions are smooth of rel. dimension 0.*
4. *If $f \circ g$ is smooth and g is unramified, then f is smooth.*

Remark 1.42 (Relation to étale morphisms)**.**

- étale $\Leftrightarrow$ flat, unramified and locally of finite presentation $\Leftrightarrow$ smooth of rel. dim. 0.
- Let $f : X \to Y$ be of locally finite presentation. Then f is smooth of rel dim. d in $x \in X$ if there exists a commutative diagram

$$\begin{array}{ccc} x \in U & \xrightarrow{\text{étale}} & \mathbb{A}^d_V \\ & {\scriptstyle f}\searrow \quad \swarrow & \\ & f(x) \in V. & \end{array}$$

Example 1.43. Let S be a scheme.

- The canonical morphisms $\mathbb{A}^n_S \to S$ and $\mathbb{P}^n_S \to S$ are smooth of rel. dim. n.
- $S = \operatorname{Spec}(k)$, $k \subseteq \overline{k}$, $\operatorname{char}(k) \neq 2$, $f \in k[T]$, $X = V(U^2 - f(T)) \subseteq \mathbb{A}^2_k = \operatorname{Spec}(K[T, U])$. Then X is smooth iff f is separable.
- $X = \operatorname{Spec}(\mathbb{Z}_p[U, V]/(U^2 - V^3 - p))$ is regular, but $X \to \operatorname{Spec}(\mathbb{Z}_p)$ is not smooth.

Lemma 1.44. *Let X, Y be k-schemes and locally of finite type. Let $x \in X, y \in Y$ be points and $\phi : \mathcal{O}_{X,x} \xrightarrow{\cong} \mathcal{O}_{Y,y}$ an isomorphism of k-algebras.*

Then there exist open neighbourhoods $x \in U \subseteq X$, $y \in V \subseteq Y$ and an isomorphism $f : U \xrightarrow{\cong} V$ such that $f(x) = y$ and $f^\#_x = \phi^{-1}$.

Proposition 1.45. *Let X/k be an integral scheme of finite type and dimension d, and let $K(X)/k$ be separable (to see what this is supposed to mean, have a look at the proof).*

Then there exists an open and dense subset $U \subseteq X$ and an isomorphism

$$U \cong \operatorname{Spec}(k[T_1, \ldots, T_d, T]/(g))$$

where $g \in k(T_1, \ldots, T_d)[T]$ is a separable irreducible monic polynomial with coefficients in $k[T_1, \ldots, T_d]$.

Proof. Find $T_1, \ldots, T_d \in K(X)$ algebraically independent and such that

$$k \hookrightarrow L := k(T_1, \ldots, T_d) \xhookrightarrow{\text{alg.\& sep.}} K(X)$$

Write $K(X) = L(\alpha)$ and let g be the minimal polynomial of α over L. After suitable multliplications, we can assume $g \in k[T_1, \ldots, T_d][T]$. Then

$$\mathcal{O}_{X,\eta} = K(X) \cong K(k[T - 1, \ldots, T_d][T]/(g)) = \mathcal{O}_{Y,(0)}$$

and the proposition follows from Lemma 1.44. $\square$

Proposition 1.46. *Let $\emptyset \neq X$ be geometrically reducible and locally of finite type over k.*

Then $X_{sm} := \{x \in X \mid X \to k \text{ is smooth in } x\} \subseteq X$ is open and dense.

Proof. The openness was stated in Proposition 1.41. It suffices to show: for any irreducible component Z of X there exists an $\emptyset \neq U = \operatorname{Spec}(A) \subseteq X$ affine and open such that $U \subseteq Z$ and $U_{\text{sm}} = X_{\text{sm}} \cap U$ is dense in U.

X is locally noetherian, therefore X locally has only finitely many irreducible components. Therefore, for $U \subset Z$ open the set $U \setminus \bigcup_{Z' \neq Z \text{irred. comp.}} (U \cap Z')$ is open in X and wlog we can assume X to be integral.

Using 1.45 and 1.18, we can assume $X = \operatorname{Spec}(k[T_1, \dots, T_d, T]/(g))$ with g separable and irreducible. Because g is separable, we have $\frac{\partial g}{\partial T} \neq 0$. Since X is reduced, this implies that $X_{\text{sm}} = \{x \in X \mid \exists i \in \{1, \dots, d, \emptyset\} \frac{\partial g}{\partial T_i}(x) \neq 0\} \neq \emptyset$ is non-empty and therefore dense. □

1.3 Group schemes over a field

Let k be a field and $S = \operatorname{Spec} k$.

Lemma 1.47. *Let G be a group scheme over k. Then $G \to \operatorname{Spec} k$ is separated.*

Proof. Let $\pi\colon G \to S$ the structure morphism. Then π is separated if and only if $e\colon S \to G$ is a closed immersion. For any $x \in \operatorname{im}(e) \in G$, choose an affine open neighbourhood $x \in U = \operatorname{Spec} A \subseteq G$. Then $\pi|_U \circ e = \operatorname{id}_S$, hence the induced map $A \xrightarrow{\Gamma(e)} k$ has a section $\Gamma(\pi|_U)$ and is therefore surjective. Thus e is a closed immersion. □

Proposition 1.48. *Let G be a group scheme locally of finite type over k. Then G is smooth over k if and only if G is geometrically reduced.*

Proof. The first direction is immediate, since smoothness is invariant under base change and smooth over a field implies reduced. Conversely, for any field extension ℓ/k by a prior result G is smooth over k if and only if G is smooth over ℓ. Thus we may assume $k = \bar{k}$. By **??** and **??**, we obtain $G_{\text{sm}} \neq \emptyset$. By the transitive action of $G(k)$ on G, every closed point is smooth. Since

$$G_{(0)} = \{g \in G \mid \dim \overline{\{g\}} = 0\}$$

is very dense in G and $G_{\text{sm}} \subseteq G$ is open, the result follows. □

Lemma 1.49. *Let k be perfect and G a group scheme locally of finite type over k. Then the induced reduced subscheme G_{red} is a subgroup scheme of G.*

Proof. Since $(-)_{\text{red}}$ is a functor, we obtain $i\colon G_{\text{red}} \to G_{\text{red}}$ and $e\colon S \to G_{\text{red}}$. By **??**, reduced is equivalent to geometrically reduced since k is perfect. Thus $G_{\text{red}} \times_k G_{\text{red}}$ is reduced and we obtain

$$\begin{array}{ccc} Gx_kG & \xrightarrow{m} & G \\ \uparrow & & \uparrow \\ G_{\text{red}} \times_k G_{\text{red}} & \dashrightarrow & G_{\text{red}} \end{array}.$$

□

Corollary 1.50. *If k is perfect and G a group scheme locally of finite type over k. Then G_{red} is smooth over k.*

Lemma 1.51. *Let G be locally of finite type over k. Then G is geometrically irreducible if (and only if) G is connected.*

Proof. Since $G(k) \neq \emptyset$, we have a morphism $\operatorname{Spec} k \to G$ and $\operatorname{Spec} k$ is geometrically connected. Thus G is geometrically connected. We may therefore assume $k = \bar{k}$. Since the statement is purely topological, we may further assume that G is reduced and thus smooth over k. Hence G is regular by **??**, in particular for every $g \in G$ the local ring $\mathcal{O}_{G,g}$ is regular and hence an integral domain. Since G is locally noetherian and connected, the claim follows. □

Definition 1.52. An *abelian variety* over k is a connected, geometrically reduced and proper k-group scheme.

Remark 1.53. Abelian varieties are smooth and geometrically integral.

Example 1.54. Elliptic curves are abelian varieties of dimension 1.

The goal is now to show that abelian varieties are commutative group schemes.

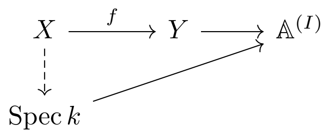
Lemma 1.55. *Let X be a proper, geometrically connected and geometrically reduced k-scheme and Y an affine k-scheme. Then every morphism $X \xrightarrow{f} Y$ factors over a k-valued point of Y.*

Proof. By the Liouville theorem for schemes, the global sections of $\mathcal{O}_{X_{\bar{k}}}$ is $\bar{k}$. Since $k \to \bar{k}$ is flat, we obtain

$$\Gamma(X, \mathcal{O}_X) \otimes_k \bar{k} \xrightarrow{\simeq} \Gamma(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}).$$

Since $k \to \bar{k}$ is even faithfully flat, we obtain $\Gamma(X, \mathcal{O}_X) \simeq k$.

Choose an embedding $Y \hookrightarrow \mathbb{A}_k^{(I)}$. Then a morphism $f\colon X \to Y$ is equivalent to a morphism $X \xrightarrow{f} Y \hookrightarrow \mathbb{A}_k^{(I)}$, which is equivalent to the datum of a family of $e_i \in \Gamma(X, \mathcal{O}_X)$ which corresponds to a morphism $\operatorname{Spec} k \xrightarrow{e} \mathbb{A}_k^{(I)}$. Thus by construction we obtain a factorisation



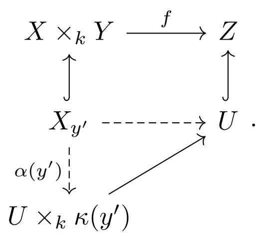
where the dashed arrow is induced from the isomorphism $\Gamma(X, \mathcal{O}_X) \simeq k$. □

Lemma 1.56 (Rigidity)**.** *Let X be a geometrically reduced, geometrically connected and proper k-scheme with $X(k) \neq \emptyset$. Let further Y be an integral scheme over k, Z be a separated k-scheme and $f\colon X \times_k Y \to Z$ a morphism such that there exists $y \in Y(k)$ such that $f|_{X_y}$ factors via a k-point $z \in Z(k)$. Then f factors via pr_2.*

Proof. Consider the composition

$$g\colon X \times_k Y \xrightarrow{pr_2} Y \simeq \operatorname{Spec} k \times_k Y \xrightarrow{(x_0, \mathrm{id})} X \times_k Y \xrightarrow{f} Z$$

where x_0 is an arbitrarily chosen k-rational point of X. It remains to show that $f = g$. Choose an open affine neighbourhood $z \in U \subseteq Z$. Then $X_y = \mathrm{pr}_2^{-1}(y) \subseteq f^{-1}(U)$. Since X is proper, pr_2 is a closed map. Thus there exists a $y \in V \subseteq Y$ open with $\mathrm{pr}_2^{-1}(V) \subseteq f^{-1}(U)$. For any $y' \in V$, we obtain



By 1.55, the morphism $\alpha(y')$ factors over a $\kappa(y')$-valued point. Thus f and g agree on the dense open subset $X \times_k V$. By reduced-to-separated, the result follows. □

Corollary 1.57. *Let A and B be abelian varieties over k and f a morphism of k-schemes $A \to B$. If under the induced map $f(k)\colon A(k) \to B(k)$ the identity e_A is mapped to e_B.*

Proof. Consider the composition

$$g\colon A \times_k A \xrightarrow{(f \circ m_A)\times(i_B \circ m_A \circ (f\times f))} B \times_k B \xrightarrow{m_B} B.$$

It remains to show that the image of g is precisely $\{e_B\}$. By assumption $f(e_A) = e_B$ and thus

$$g(\{e_A\} \times_k A) = \{e_B\} = g(A \times_k \{e_A\}).$$

By repeated application of 1.56, g factors via pr_1 and pr_2. Thus g is constant and e_B is in the image. □

Corollary 1.58. *Every abelian variety is commutative.*

Proof. Apply 1.57 on $i\colon A \to A$. □

Lemma 1.59. *Let X be a connected scheme over k and Y a geometrically connected scheme over k. If $\mathrm{Hom}_k(Y, X) \neq \emptyset$, then X is geometrically connected.*

Proof. Use that $X_{\bar{k}} \to X$ is an open and closed immersion. Let $\emptyset \neq Z \subseteq X_{\bar{k}}$ be open and closed. Consider the commutative diagram

$$\begin{array}{ccccc} \bar{f}^{-1}(Z) = Z \times_k Y & \longrightarrow & Y_{\bar{k}} & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle f} \\ Z & \hookrightarrow & X_{\bar{k}} & \xrightarrow{\pi} & X \end{array}.$$

We obtain $\bar{f}^{-1}(Z) = Y_{\bar{k}}$. Set $Z' = Y_{\bar{k}} \setminus Z$. If Z' is not-empty, then by the same argument $\bar{f}^{-1}(Z') = Y_{\bar{k}}$. Contradiction. □

Proposition 1.60. *Let G be a group scheme locally of finite type over k.*

1. *If $U, V \subseteq G$ are open and dense. Then $UV = G$ as topological spaces.*
2. *If G is irreducible, then G is quasi-compact.*
3. *Any subgroupscheme $H \subseteq G$ is a closed subscheme.*

Proof. We reduce to $k = \bar{k}$.

1. We know that $G_{\bar{k}} \to G$ is an open and closed immersion. Taking pre-images then preserves open and dense (???) and the result follows.
2. By **??** G is geometrically irreducible and $G_{\bar{k}} \to G$ is surjective, i.e. the quasi-compactness of $G_{\bar{k}}$ implies the quasi-compactness of G. ´
3. By **??**, being a closed immersion can be tested by faithfully flat descent.

Now suppose $k = \bar{k}$.

1. It suffices to show that $U(k)V(k) = G(k)$, since $\overline{U(k)V(k)}$ is very dense in $\overline{UV}$. Since $i\colon G \to G$ is an isomorphism of schemes, $V(k)^{-1} \subseteq G(k)$ is open and dense. Thus for all $g \in G$, $g(V(k)^{-1})$ is open and dense. Thus there exists $u \in g(V(k)^{-1}) \cap U(k)$, i.e. there exists $v \in V(k)$ such that $gv^{-1} = u$, i.e. $g = uv$.
2. Let $U \subseteq G$ be open, dense and quasi-compact. Then $U \times_k U$ is quasi-compact and $G = \mathrm{im}(U \times_k U \to G)$ is quasi-compact.

3. Put the induced reduced subscheme structure on $\bar{H} \subseteq G$. By **??**, the maps $H \to \operatorname{Spec} k$ and $\bar{H} \to \operatorname{Spec} k$ are universally open. Since $H \subseteq \bar{H}$ is dense, we obtain

$$H \times_k H \subseteq H \times_k \bar{H} \subseteq \bar{H} \times_k \bar{H}$$

is dense. Since $H \times_k H \subseteq m^{-1}(H) \subseteq m^{-1}(\bar{H}) \hookrightarrow G \times G$, we obtain topologically $\bar{H} \times \bar{H} \subseteq m^{-1}(\bar{H})$. Since the objects in the lower row are reduced, we therefore obtain a factorisation

$$\begin{array}{ccc} G \times G & \longrightarrow & G \\ \uparrow & & \uparrow \\ \bar{H} \times_k \bar{H} & \dashrightarrow & \bar{H} \end{array}.$$

Thus $\bar{H} \subseteq G$ is a subgroupscheme. Thus $H = H \times H = \bar{H}$ where the last equality follows from 1.

□

Definition 1.61. Let G be a group scheme locally of finite type over k and $e\colon \operatorname{Spec} k \to G$ is the unit. Then denote by G^0 the connected component of G that contains $\operatorname{im}(e)$. We call G^0 the *unit component* of G.

Remark 1.62. Since G is locally noetherian, G^0 is open and closed.

Proposition 1.63. *Let G be a group scheme locally of finite type over k.*

1. *G^0 is a quasi-compact, geometrically-irreducible and normal subgroupscheme of G.*

2. *Any group morphism $G \to H$ with H locally of finite type over k induces a group homomorphism $G^0 \to H^0$.*

3. *For any field extension ℓ/k, we have*

$$(G \times_k \ell)^0 = G^0 \times_k \ell.$$

Proof. 1. Since G^0 is connected and contains a k-rational point, by **??** G^0 is geometrically connected. Then $G_0 \times_k G_0$ is connected and

$$\begin{array}{ccc} G \times_k G & \longrightarrow & G \\ \uparrow & & \uparrow \\ G^0 \times_k G^0 & \dashrightarrow & G^0 \end{array}.$$

Since $G^0 \hookrightarrow G \xrightarrow{i} G$ factors over $G^0 \hookrightarrow G$, G^0 is a subgroupscheme. By **??**, G^0 is geometrically irreducible and therefore by **??** it is quasi-compact. For normality consider a connected component G' of G. Then we have a commutative diagram

$$\begin{array}{ccc} G \times_k G & \xrightarrow{(g,h)\mapsto ghg^{-1}} & G \\ \uparrow & & \uparrow \\ G' \times_k G^0 & \dashrightarrow & G^0 \end{array}.$$

Since $G' \times G^0$ is connected, the image of the upper horizontal arrow is in G^0.

2. Any group homomorphism sends the identity to the identity, i.e. the composition $G^0 \hookrightarrow G \to H$ factors via $H^0 \hookrightarrow H$.

3. Since G^0 is geometrically connected, the scheme $G^0 \times_k \ell$ is connected. Moreover $G^0 \times_k \ell \subseteq G \times_k \ell$ is open and closed. Finally, the identity of $G \times_k \ell$ is contained in $G^0 \times_k \ell$ by the universal property of the fibre product.

□

The proof of the following lemma is left as an exercise to the reader.

Lemma 1.64. *Let G be a group scheme locally of finite type over k. Then every connected component of G is quasi-compact and geometrically irreducible and G is equidimensional.*

Proposition 1.65. *Let $f: G \to H$ be a group homomorphism of group schemes locally of finite type over k. Then*

1. $\mathrm{im}(f) \subseteq H$ *is closed.*
2. $\dim(G) = \dim(\mathrm{im}(f)) + \dim(\ker(f))$.
3. *Is H smooth over k and f surjective, then f is faithfully flat.*

Remark 1.66. For any integral morphism $f: X \to Y$ and $Z \subseteq X$ closed the image $f(Z)$ is closed in Y and $\dim(Z) = \dim(f(Z))$.

Proof. Since $H_{\bar{k}} \xrightarrow{\pi} H$ is integral and surjective and $\dim(Z) = \dim(\pi(Z))$ for any closed subset $Z \subseteq H_{\bar{k}}$, we may assume $k = \bar{k}$.

3. Since smooth implies reduced, H^0 is reduced and by **??** H^0 is irreducible. Thus H^0 is integral. By generic flatness, we have a $V \subseteq H^0$ that is open and dense such that $f^{-1}(V) \to V$ is flat. Thus for all $h \in H(k)$, the map $f^{-1}(hV) \xrightarrow{f} hV$ is flat. By covering H with translates of V, we obtain f is flat.

1. We may assume that G is reduced and thus G is smooth over k by **??**. Let C be $C_{\mathrm{red}} = \overline{f(G)}^H$. We claim that C is a subgroupscheme of H. Then $G \to C$ is quasi-compact and dominant. Thus we have a factorisation

$$\begin{array}{ccccc} G \times_k G & \longrightarrow & C \times_k C & \longrightarrow & H \times_k H \\ \downarrow m_G & & \downarrow m_C & & \downarrow m_H \\ G & \xrightarrow{f} & C & \hookrightarrow & H \end{array}.$$

Analogously one obtains

$$\begin{array}{ccc} C & \dashrightarrow & C \\ \downarrow & & \downarrow \\ H & \longrightarrow & H \end{array}.$$

Thus we may assume that f is dominant. By the theorem of Chevalley, $f(G)$ is constructible and is therefore dense. Hence there exists an open $U \subseteq H$ such that $U \subseteq f(G)$. Thus $H = U \cdot U \subseteq f(G)$ and $f(G) = H$ is closed.

2. We may assume that also H is reduced and that $f(G) = H$. Then H is smooth over k and f is flat. By **??** we have $f(G^0) \subseteq H$ is open and by 1) also closed. Thus $G^0 \xrightarrow{f} H^0$ is surjective. We have $\dim(G^0) = \dim(G)$, $\dim(H^0) = \dim(H)$ and $\dim(\ker(f^0)) = \dim(\ker(f)^0)$. Now the result follows since all fibres are isomorphic and dimension is additive under flat morphism in non-empty fibres ([**?**] 14.119).

□

Lemma 1.67. *Let X/k be of locally finite type. Then X_{sm} is constructible.*

Proof. WLOG X is affine. Then the assertion follows from B53, B72c and OCG13Z. □

Proposition 1.68. *If $f : X \to Y$ is a morphism of schemes and $|Y|$ is discrete, then f is universally open. (cf. Corollary* **??***)*

Proof. Universal openness is local on the target, therefore wlog $\#Y = 1$. Since, in addition, universal openness is a topological condition, we can assume Y to be reduced. Therefore let $Y = \operatorname{Spec} k$ for k a field.

Let $Y' \to Y$ be arbitrary. Since openness is local on the domain, assume $X = \operatorname{Spec} A$; $Y' = \operatorname{Spec} B$ and therefore $X \times_Y Y' = \operatorname{Spec} A \otimes_k B$. Write $A = \operatorname{colim}_\alpha A_\alpha$ as colimit over the finitely generated subalgebras $A_\alpha \subseteq A$. Then

$$A \otimes_k B = \operatorname{colim}_\alpha (A_\alpha \otimes_k B)\,.$$

Let $t \in B$ and denote by f' the base change of f. We show that $U = f'(D(t))$ is open in $\operatorname{Spec} B$. Let $t \in A_\alpha \otimes_k B$ for suitable α. Applying Corollary 1.23 shows that $f'' : \operatorname{Spec} A_\alpha \otimes_k B \to \operatorname{Spec} B$ is open. Therefore $U' = f''(D(t)) \subseteq \operatorname{Spec} B$ is open, so it suffices to check $U = U'$.

We have $U \subseteq U''$ by assumption.Let $y \in U'$. It suffices to show

$$(f')^{-1}(y) \cap D(t) \neq \emptyset\,.$$

But $(f')^{-1} = g^{-1}(W)$ with $g : (f')^{-1}(y) = \operatorname{Spec}(B' \otimes_B \kappa(y)), (f'')^{-1}(y) = \operatorname{Spec}(A_\alpha \otimes_k \kappa(y))$ and $W = (f'')^{-1}(y) \cap D(t)$. Since $\kappa(y)/k$ is flat, we have an injection $A_\alpha \otimes_k \kappa(y) \hookrightarrow A_\alpha \otimes_k \kappa(y)$ and g is dominant. This implies $g^{-1}(W) \neq \emptyset$, since W is open and non empty. □

1.4 Differentials and Smoothness

Definition 1.69. $A \to B$, $M \in B$-Mod. Define

$$\operatorname{Der}_A(B, M) = \{D \in operatornameHom_A(B, M) \mid D(Fg) = fD(g) + gD(f) \quad \forall f, g \in B\}\,.$$

The module of Kähler differentials of B/A is a pair $(\Omega^1_{B/A}, d_{B/A})$ with $\Omega^1_{B/A} \in B$-Mod, $d_{B/A} \in \operatorname{Der}_A(B, \Omega^1_{B/A})$ such that $d_{B/A,*} : \operatorname{Hom}_B(\Omega^1_{B/A}, M) \xrightarrow{\cong} \operatorname{Der}_A(B, M)$ is an isomorphism.

Lemma 1.70. *For $A \to B$, we have*

$$\Omega^1_{B/A} \cong \bigoplus_{b \in B} dbB \Big/ \Big\langle \begin{matrix} & d(bb') = dbdb' + b'db \\ d(b+b') = db + db'\,, & da = 0 \end{matrix} \mid b, b' \in B, a \in A \Big\rangle\,.$$

The universal differential $d_{B/A}$ is given by $b \mapsto [db]$. For $I = \ker(B \otimes_A B \to B)$, we have an isomorphism

$$\Omega^1_{B/A} \to I/I^2\,, \quad [db] \mapsto \overline{1 \otimes b - b \otimes 1}\,.$$

Proof. This formal calculation. □

Example 1.71. 1. Let $B = A[T_1, \dots, T_n]$. Then $\Omega^1_{B/A} = \bigoplus_{i=1}^n dT_i B$.

2. Let L/K be an separable extension. Then $\Omega^1_{L/K} = 0$.

Lemma 1.72. *Let A', B be A-algebras and $S \subseteq A$ multiplicatively closed. Then we have $S^{-1}\Omega^1_{B/A} = \Omega^1 S^{-1}B/A$ and $\Omega^1 B/A \otimes_B B' = \Omega^1 B'/A'$.*

Lemma 1.73. *$f : A \to B, g : B \to C$. Then we have an exact sequence*

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

of C-modules. If g is surjective with kernel I, the sequence

$$I/I^2 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0$$

is exact.

Example 1.74. Let A be a ring and $B = A[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$. Then

$$\Omega^1_{B/A} \cong \bigoplus_{i=1}^{n} dT_i B / \langle df_i \mid i = 1, \ldots, n \rangle ,$$

where $df_i = \sum_k \frac{\partial f_i}{\partial T_k} T_k$.

Definition 1.75. Let $i : Y \hookrightarrow X$ be an immersion $Y \xrightarrow{\text{closed}} U \xrightarrow{\text{open}} X$ and I the associated ideal sheaf. Then define $\omega_{Y/X} = I/I^2$ as an $\mathcal{O}_Y$-module.

For $f : X \to S$ the module of Kähler differentials is given by

$$\Omega^1_{X/S} := \omega_X / X \times_S X$$

with $\Delta : X \to X \times_S X$.

Remark 1.76. $X \to S$ mono implies $\Omega^1_{X/S} = 0$.

Proposition 1.77. *1. For $X \xrightarrow{f} Y \to S$ we have an exact sequence*

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0 .$$

2. Given $X \to S \leftarrow Y$, we have an isomorphism

$$p_1^*\Omega^1_{X/S} \oplus p_2^*\Omega^1_{Y/S} \cong \Omega^1_{X \times_S Y} .$$

3. Given $Z \xrightarrow{closed} X \to S$, we have

$$\omega_{Z/Y} \to i^*\Omega^1_{X/S} \to \Omega^1_{Z/S} \to 0 .$$

4. If $X \to S$ is of locally finite type, then $\Omega^1_{X/S}$ is of finite presentation.

Proposition 1.78 ([?], 18.64)**.** *Let k be a field, X/k of locally finite type and $x \in X$.*

Then X is smooth in X iff $\Omega^1_{X/S}$ is a free $\mathcal{O}_{X,x}$-module of dimension $\dim X$.

Proposition 1.79. *Let $X' = X \times_S \times S'$ be cartesian. Then there exists a canonical isomorphism*

$$h^*\Omega^1_{X/S} \xrightarrow{\cong} \Omega^1_{X/S} .$$

Proof. exercise sheet numero sei □

Proposition 1.80. *Let $\pi : G \to S$ be a group scheme with unit $e \in G(S)$. Then there are the following isomorphisms of $\mathcal{O}_G$-modules*

$$\Omega^1_{G/S} \cong \pi^* e^* \Omega^1_{G/S} \cong \pi^* \underbrace{\omega_{S/G}}_{via\ e} .$$

Proof. First, consider the cartesian diagram

$$\begin{array}{ccc} G \times_S G & \xrightarrow{m} & G \\ \downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \pi} \\ G & \xrightarrow{\pi} & S \end{array}$$

for $i = 1, 2$. It yields

$$m^*\Omega^1_{G/S} \cong \Omega^1_{G \times_S G/G} \cong p_i^*\Omega^1_{G/S} .$$

Consider also $i = (e\pi, \mathrm{id}_G) : G \to G \times_S G$. Then

$$\Omega^1_{G/S} \cong \mathrm{id}_G^* \Omega^1_{G/S} = i^* p_2^* \Omega^1_{G/S} = i^* m^* \Omega^1_{G/S} = i^* p_1^* \Omega^1_{G/S} = (e\pi)^* \Omega^1_{G/S} = \pi^* e^* \Omega^1_{G/S} .$$

Secondly, consider the diagram of sections

$$\begin{array}{ccc} S & \xrightarrow{e} & G \\ {\scriptstyle \pi}\uparrow\downarrow{\scriptstyle e} & & {\scriptstyle p_1}\uparrow\downarrow{\scriptstyle \Delta} \\ G & \xrightarrow{i} & G \times_S G , \end{array}$$

where $i = (e\pi, \mathrm{id}_G)$. We deduce $e^*\Omega^1_{G/S} \cong \omega_e$ and π^* yields $\pi^* e^* \Omega^1_{G/S} \cong \pi^* \omega_e$. □

1.5 Functor of points

Reference: Demargue-Gabriel: Groups algebraiguexue

We want to define quotient group schemes. For scheme S, S-subgroup $H \hookrightarrow G$ we want a short exact sequence

$$0 \to H(T) \to G(T) \to (G/H)(T) \to 0 .$$

But the presheaf G/H is not generally a sheaf.

As an ansatz, consider the yoneda embedding

$$y : \mathrm{Sch}_S \hookrightarrow \mathrm{PSh}(\mathrm{Sch}_S) .$$

Grothendieck showed: the fpqc-topology is subcanonical, ie. presentable presheaves are sheaves in the fpqc-topology. Therefore, it may be useful to consider the fppf-sheafification of $(T \mapsto G(T)/H(T))$. (why fppf and not fpqc: later)

Remark 1.81. Let be:
LRS = category of locally ringed spaces
Sch = category of schemes
Aff = category of affine schemes
All of these are full subcats of each other.
For

$$y : \mathrm{Sch}_S \hookrightarrow \mathrm{PSh}(\mathrm{Sch}_S) ,$$

our goal is to consider the essential image of y without reference to Sch_S

Remark 1.82 (Ansatz)**.** Schemes are build from affine schemes through glueing on open immersions, i.e. every scheme is a colimit (coequalizer) of affine schemes and open immersions

$$\coprod_{i,j \in I} \coprod_{k \in J} \mathrm{Spec}(B_{i,j}) \rightrightarrows \coprod_{i \in I} \mathrm{Spec}(A_i) \to X .$$

Example 1.83. For scheme X and open subschemes $U, V \subseteq X$ with $U \cup V = X$ the canonical map

$$y(U) \amalg_{y(W)} y(V) \to y(X)$$

is not generally an isomorphism in PSh(Sch) for $W = U \cap V$.

But after sheafification $a : \mathrm{PSh}(\mathrm{Sch}) \to \mathrm{Sh}(\mathrm{Sch})$ we get two isomorphisms (since presheaves are reflective subcat of sheaves)

$$a(y(U) \amalg_{y(W)} y(V)) \to a(y(U)) \amalg_{a(y(W))} a(y(V)) \to a(y(X)) = y(X) .$$

Question: What is an open immersion between affine schemes?

Proposition 1.84. *Let $f : X \to Y$ be a morphism of schemes. TFAE (?)*

(i) f is open immersion,

(ii) f is étale,

(iii) f is flat mono of locally finite presentation.

Remark 1.85. A family $\{\mathrm{Spec}(A_i) \to \mathrm{Spec}(A)\}_{i \in I}$ of open immersions is an open covering iff it can be verfeinert by an open covering of the form $\{\mathrm{Spec}(A_{f_\alpha}) \to \mathrm{Spec}(A)\}_\alpha$. TFAE

1. $\{\mathrm{Spec}(A_{f_\alpha}) \to \mathrm{Spec}(A)\}_\alpha$ is an open covering,
2. $A \to \prod_\alpha A_{f_\alpha}$ is faithfully flat,
3. $(1) = (f_\alpha \mid \alpha) \subseteq A$.

Sh(Sch) $\simeq$ Sh(Aff) and there we only need open immersions:

Remark 1.86. Following the preceding remark, we have the sites $\mathrm{Aff}^{\mathrm{Zar}}$ of affine schmes with zariksi topology.

The restriction PSh(Sch) $\to$ PSh(Aff) induces an equivalence of cats Sh(Sch) $\to$ Sh(Aff) by the comparison lemma. A quasi inverse is given by

$$F \mapsto \hat{F} : X \mapsto \lim_{\mathrm{Spec}\, A \to X} F(\mathrm{Spec}\, A)\,.$$

The essential image of

$$\mathrm{Sch} \overset{y}{\hookrightarrow} \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Sch}) \xrightarrow{i^*} \mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Aff})$$

consists exactly of those sheaves that can be written as coequalizer of a diagram of the form

$$\coprod_{i,j,k} \mathrm{Spec}(A_{ijk}) \rightrightarrows \coprod_i \mathrm{Spec}\, A_i$$

The problem is to check whether something is or is not a sheaf in $\mathrm{Sh}_{\mathrm{Zar}}(\mathrm{Aff})$.

Remark 1.87. Let PSh denote PSh(Aff) = Fun(CRing, Set) and Aff = $\mathrm{im}(y : \mathrm{CRing}^{\mathrm{op}} \to \mathrm{PSh})$. Set $S(R) := y(R)$ as an "affine scheme".

For A ring, $X \in$ PSh, $p \in X(A)$ we have

$$p^\# : S(A) \to X$$

in PSh via $S(A)(R) \to X(R), \phi \mapsto X(\phi)(R)$.

A subfunctor $U \hookrightarrow X$ in PSh is an equivalence class of monos in PSh.

When is a subfunctor an open immersion? "Open immersion" is local on target, so we can check this on open coverings by affine schemes. These are of the shortly following form

Remark 1.88. For an ideal $I \subseteq A$ we have decompostion $V(I) \to \mathrm{Spec}(A) \leftarrow D(I)$. For ringmorphism $\phi : A \to R$ have

$$\mathrm{Spec}(\phi)^{-1}(D(I) = D(\phi(I) \cdot R))\,.$$

Therefore

$$\begin{array}{ccc} \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}(A) \\ & \overset{\exists !}{\searrow} & \downarrow \\ & & D(I) \end{array}$$

factorizes iff $\phi(I) \cdot R = R$.

Definition 1.89.
- For ideal $I \subseteq A$ define subfunctor $S(A)_I \subseteq S(A)$ via

$$S(A)_I(R). = \{\phi : A \to R \mid \phi(I) \cdot R = R\}$$

(by the above lemma, these are precisely the points which come from the open subscheme $D(I)$)

- An *open subfunctor* is a subfunctor $U \hookrightarrow X$ such that for every morphism $S(A) \to X$ the projection map from the (pointwise) fibre product $U \times_X S(A)$ to a subfunctor of the form $S(A)_I$ is an isomorphism for a suitable ideal $I \subseteq A$.

Example 1.90. For $p \neq q$ prime numbers we have $X = \operatorname{Spec} \mathbb{Z} = D(p) \cup D(q)$. But

$$X(\mathbb{Z}) \supsetneq D(p)(\mathbb{Z}) \cup D(q)(\mathbb{Z}),$$

since both sets on the right are empty.

So this isn't quite right either. Maybe fields?

Definition 1.91.
- A family $(U_i \hookrightarrow X)_{i\in I}$ of open immersions in PSh is an *open covering*, if for every field k we have

$$X(k) = \bigcup_{i\in I} U_i(k).$$

(here we could replace "field" by "local ring". The idea is that "points" are specs of fields, and we shouldnt require these covering conditions for all objects, just for points)

- For ring A a *partition of unity* is given by a finite family (f_i, x_i) with $f_i, x_i \in A$ such that

$$\sum_i f_i x_i = 1.$$

In this case $(S(A)_{(f_i)})_{i\in I}$ is an open covering of $S(A)$.

- A presheaf $X \in \mathrm{PSh}$ is *local* (ie. it is a *sheaf*) if for all $A \in \mathrm{CRing}$ and all partitions of unity in A the induced diagram

$$X(A) \to \textstyle\prod_i X(A_{f_i}) \rightrightarrows \prod_{i,j} X(Af_i f_j)$$

is a limit-diagram.

- A *scheme* is a local presheaf that allows an open covering by affine schemes.

Remark 1.92. An open subfunctor of a scheme is a scheme.

Remark 1.93.
- Let $X = (|X|, \mathcal{O}_X)$ be an locally ringed space. Obtain $S(X) \in \mathrm{PSh}$ with $S(X)(R) = \mathrm{Hom}_{\mathrm{LRS}}(\mathrm{Spec}(R), X)$
- For $U \subseteq X$ open $S(U) \subseteq S(X)$ open subfunctor.
- A covering $X = \bigcup_{i\in I} U_I$ yields an open covering $(S(U_i))_i$ of $S(X)$.

Proposition 1.94. *A locally ringed space X is a scheme iff $S(X)$ is a scheme.*

Remark 1.95. The Vorschrift $X \mapsto S(X)$ defines a functor LRS $\to$ PSh that has a left adjoint.

§7 fppf-topology and algebraic spaces

Let S be a scheme and Sch_S the category of S-Schemes. We have the following problem. Given an S-group scheme G and an S-sub-group-scheme H the quotient G/H may not exist in the category of S-schemes.
We remedy this by forming the quotient in a larger category, namely the category of algberaic spaces or fppf-sheaves, and then study conditions under which the constructed quotient lives in Sch_S.

Remark 1.96. Given a scheme S we have the following topologies on Sch_S

$$\text{Zariski} \subset \text{étale} \subset \text{fppf} \subset \text{fpqc}.$$

Theorem 1.97 (Grothendieck, 023Q Stacks)**.** *Let S be a scheme. Then the representable* $\hom_{Sch}(-, S)$ *is a fpqc-sheaf.*

Remark 1.98.

- If G/H as above is a scheme its yoneda image must be an fpqc-sheaf.
- However sheafification does not work for fpqc-presheaves.

Definition 1.99. We call a family of morphisms $\{f_i : X_i \to X\}$ of schemes an fppf-covering iff

1) f_i are flat and of finite presentation $\forall i$ and
2) they are jointly surjective, i.e. $X = \bigcup_i f_i(X_i)$.

Remark 1.100. A preseheaf $F : Sch_S^{op} \to Grp$ is an fppf-sheaf iff the associated set-valued presheaf is an fppf-sheaf, since the forgetful functor from groups to sets commutes with limits (as it has an adjoint).

Exactness of a sequence of sheaves can be checked just as for topological spaces

Remark 1.101. Let τ be a topology on the site Sch_S. And let $0 \to F \to G \to H$ be a sequence of sheaves with values in abelian groups/modules/.... It is exact iff

1) for all $X \in Sch_S$ the sequence $0 \to F(X) \to G(X) \to H(X)$ is exact and
2) For all $X \in Sch_S$ and all $h \in H(X)$ there is a covering $\{X_i \to X\}$ s.t. $h|_{X_i}$ is in the image of $G(X_i) \to H(X_i)$.

Example 1.102. Let $n \geq 1$ be an integer. The sequence $0 \to \mu_{n,S} \to \mathbb{G}_{m,S} \overset{(-)^n}{\to} \mathbb{G}_{m,S}$ is an exact sequence of presheaves on Sch_S. If $n \in \mathcal{O}_S(S)^\times$, then $(-)^n$ is surjective w.r.t. to the étale topology on Sch_S. If $n \notin \mathcal{O}_S(S)^\times$ then $(-)^n$ is not surjective w.r.t. the étale topology but w.r.t. the fppf-topology on Sch_S.

Let $\mathcal{C}$ be any category. We denote by $y : \mathcal{C} \to PSh(\mathcal{C}),\ X \mapsto y(X) = \hom_{\mathcal{C}}(-, X)$ the yoneda embedding.

Definition 1.103.

(i) Let $F, G \in PSh(Sch_S)$. We call a morphism $F \to G$ *representable* iff for all $X \in Sch_S$ and all morphisms $y(X) \to G$ the fibre product $F \times_G y(X)$ is representable.

(ii) Let furthermore $\mathbb{P}$ be a property of morphisms of schemes which is closed under pre- and postcomoposition with isomorphisms. We say that a representable morphism $F \to G$ *has* $\mathbb{P}$ iff for all $X \in Sch_S$ and all $y(X) \to G$ the morphism of schemes corresponding to

$$F \times_G y(X) \to y(X)$$

has the property $\mathbb{P}$.

Remark 1.104. Note that in the second part of the above definition the object $F \times_G y(X)$ is representable s.t. the definition becomes meaningful.

Remark 1.105. Representable morphisms are closed under composition and base change. A representable morphism with representable target has representable source, because: Let $F \to G$ be representable, $G = y(X)$. Then for $id : y(X) \to y(X)$ the object $F \times_G G = F$ is representable.

Lemma 1.106. *Let $F \in PSh(Sch_S)$. Then we have*

$$\Delta : F \to F \times_S F \text{ representable} \iff \forall X \in Sch_S : \text{every morphism } y(X) \to F \text{ is representable.}$$

Proof. $\Rightarrow$: Let $y(X) \to F$ and $y(Y) \to F$ be given as in the definition. The diagram

$$\begin{array}{ccc} y(X) \times_F y(Y) & \longrightarrow & y(X) \times_S y(Y) = y(X \times_S Y) \\ \downarrow & & \downarrow \\ F & \longrightarrow & F \times_S F \end{array}$$

is cartesian. So the upper map is representable with representable target. So the term $y(X) \times_F y(Y)$ is representable.

$\Leftarrow$: Let $y(X) \to F \times_S F$. We claim

$$F \times_{F \times_S F} y(X) = y(X) \times_{y(X) \times_S y(X)} (y(X) \times_F y(X)).$$

By assumption the term $y(X) \times_F y(X)$ is representable and since then every term on the RHS is representable so is the left term. So it remains to show the claim. For this consider the diagram (we omit the yoneda embedding from the notation)

$$\begin{array}{ccccc} X \times_{X \times_S X} (X \times_F X) & \longrightarrow & X \times_F X & \longrightarrow & F \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X \times_S X & \longrightarrow & F \times_S F. \end{array}$$

Both little square are cartesian. Thus, the outer square is cartesian which shows the claim. □

Definition 1.107. An *algebraic space (over S)* is a sheaf $X \in PSh(Sch_S)$ with respect to the fppf-topology s.t.

i) $X \to X \times_S X$ is representable and

ii) There exists an S-scheme U and a morphism $y(U) \to X$ which is surjective in the étale topology.

From this we obtain the full subcategory $AlgSpc_S \subset Sh_{fppf}(Sch_S)$.

Remark 1.108. The category of algebraice spaces is closed under fibre products in $PSh(Sch_S)$ (what does this mean?)

Lemma 1.109. *Let $Y \in AlgSpc_S$ and $X \to Y$ representable. Then $X \in AlgSpc_S$.*

Proof. The proof of this was not given completely. □

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